Fluids of Vortices And Dark Matter

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By considering full-field string solutions of the Abelian–Higgs model, we modify the model of a fluid of strings (which is composed of Nambu strings) to obtain a model for a "fluid of vortices." With this model, and following closely Soleng's proposal of a fluid of strings as the source of a Milgrom-type correction to the Newton dynamics, we determine quantitatively the modified dynamics generated by a static, spherical fluid of vortices.

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Since its development more than eighty years ago, Einstein's theory of General Relativity has enjoyed many successes in describing gravity at very different scales, from the orbit of Mercury to the prediction of black holes or the Big Bang model. The theory, however, encounters serious problems when confronted to the motion of galaxies and galaxy clusters, where it seems to imply that there is more mass than is observed. This is the so-called "missing mass" problem, to which several solutions are being investigated: first, General Relativity may have to be modified at such distances; to conform to the observations, the modified theory would then have to yield a gravitational acceleration decreasing as $1/r$. Second, the Universe may be filled with dark matter, which would act gravitationally but would not be observable. Third, non-gravitational forces may play an important rôle at very large scales. Finally, of course, it might also prove necessary to invoke a combination of several of these solutions.

There are good reasons, however, to believe that General Relativity is not the definitive theory of gravitation, in that it is not a quantum theory. Currently, string/M-theory is seen by many as the best candidate to unify the four fundamental forces, and it is well known that in such theories Einsteinian gravity must be replaced by more complicated scalar-tensor models. In low-energy M-theory, for instance, one must compactify seven of the eleven spacetime dimensions to obtain General Relativity from eleven-dimensional supergravity. Although it is widely believed that the extra dimensions only play a significant rôle at very short distances, there exist models where gravity becomes higher dimensional [1] at very large distances as well.

A rather simple, but effective, approach to the problem is to modify the Newton force at large distance. Milgrom [2] proposed to write the real gravitational acceleration **g** as a function of the Newtonian acceleration $\mathbf{g}_N$,

$$\mu\left(\frac{g}{a_0}\right)\mathbf{g} = \mathbf{g}_N, \qquad (1)$$

where a_0 is a constant, and $\mu(x) \approx 1$ for $x \ll 1$ and $\mu(x) \approx x$ for $x \gg 1$. This modified Newtonian dynamics was used to explain, without the need for any dark matter, the observed gravitational behavior of galaxies and galaxy clusters [3,4]. In particular, it yields the a constant velocity curve at large galactic radius, $V^4 \simeq MGa_0$, which implies that the true acceleration at large distances is proportional to $1/r$ rather than $1/r^2$.

The main problem with this model is that it is supported only by its phenomenological success and has no theoretical basis.

By using a model of a fluid of ordered strings, developed some time ago by Letelier [5,6], and usual General Relativity, Soleng [7] found that a perfect fluid surrounding a point mass M leads to a force

$$g = \frac{M}{r^2} + \frac{1}{\ell(\tilde{\alpha} - 2)}\left(\frac{\ell}{r}\right)^{1-2/\tilde{\alpha}}. \qquad (2)$$

where ℓ is an integration constant. The parameter $\tilde{\alpha}$ characterizes the fluid's equation of state, and remains unspecified.

In this Letter, our main goal is to introduce a new model describing a "perfect fluid of vortices," which is a relative of the well-known string fluid [6], except that we replace the Nambu strings by (thick) solutions of the full-field equations of the Abelian–Higgs model, namely Nielsen–Olesen (NO) vortices [8]. Because we choose a particle model to describe the vortices, we are able to choose more realistically the parameter $\tilde{\alpha}$ appearing in Soleng's solution, and therefore also to determine the modified dynamics of the perfect vortex fluid surrounding a mass M.

Let us begin by describing the original string fluid model introduced in [6] (see also [5]). The starting point is the observation that the natural generalization of the perfect fluid energy-momentum tensor to the case of one dimension extended objects with equations of state "tension equals energy density" is

$$T^{\mu\nu} = \rho\left(u^\mu u^\nu - \chi^\nu\chi^\mu\right) - p\left[g^{\mu\nu} - (u^\mu u^\nu - \chi^\nu\chi^\mu)\right],$$
$$= (\rho + p)\left(u^\mu u^\nu - \chi^\nu\chi^\mu\right) - pg^{\mu\nu}. \qquad (3)$$

where ρ is the string energy-density that is equal to its tension, $p > 0$ $(p < 0)$ is a transversal pressure (tension), and u^μ and χ^μ are two smooth vector fields satisfying

$$u^\mu u^\mu = -\chi^\mu\chi_\mu = 1 \qquad (4a)$$
$$u^\mu\chi_\mu = 0. \qquad (4b)$$

At any spacetime point u^μ represents the string velocity and χ^μ the string direction at each point of the string.

The metric for a static spherically symmetric spacetime is

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (5)$$

The general solution of the Einstein equations for the energy-momentum tensor (3) and metric (5) was studied in [6]. For the equation of state

$$\rho = -\tilde\alpha\, p, \quad (6)$$

it was found (for $\tilde\alpha \neq 2$) that

$$e^{2\nu} = e^{-2\lambda} = 1 - \frac{2M}{r} - \frac{\tilde\alpha}{\tilde\alpha - 2}\left(\frac{\ell}{r}\right)^{2/\tilde\alpha}, \quad (7a)$$

$$8\pi\rho = \frac{1}{r^2}\left(\frac{\ell}{r}\right)^{2/\tilde\alpha}. \quad (7b)$$

where ℓ is an integration constant. This metric represents a bunch of radially directed Nambu strings interacting transversally, in a way described by the equation of state (6).

Our aim now is to model this transversal interaction by considering a particle model of the strings in the fluid. More precisely, we consider an Abelian–Higgs model with matter Lagrangian

$$\tilde{\mathcal{L}}_{\mathrm{M}} = \left(\mathcal{D}_\mu\Phi\right)\left(\mathcal{D}^\mu\Phi\right)^\dagger - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{\lambda}{4}\left(\Phi^\dagger\Phi - \eta^2\right)^2, \quad (8)$$

where Φ is a complex Higgs field, $\mathcal{D}_\mu$ is a gauge-covariant derivative defined by $\mathcal{D}_\mu\Phi = \nabla_\mu\Phi + ieA_\mu\Phi$ and $\tilde{F}_{\mu\nu}$ is the strength field associated with the gauge potential A_μ, that is, $\tilde{F}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

We can simplify this theory by making the conventional Ansatz in cylindrical coordinates $\{t, z, R, \phi\}$

$$\Phi = \eta\, X(R)\, e^{iN\phi} \quad (9a)$$

$$A_\mu = \frac{1}{e}\left(P_\mu - \nabla_a\chi\right). \quad (9b)$$

We can simplify this further by simultaneously rescaling our coordinates and P_μ by the string's width $w_{\mathrm{H}} \equiv 1/(\sqrt{\lambda}\eta)$. This corresponds to measuring distances in string rather than Planck units, and can be most easily done by letting $w_{\mathrm{H}} = 1$ everywhere. The only parameter left in this theory is the Bogomol'nyi parameter

$$\beta \equiv \left(\frac{m_{\mathrm{Higgs}}}{m_{\mathrm{gauge}}}\right)^2 = \frac{\lambda}{2e^2}. \quad (10)$$

The NO solution is found by assuming that $N = 1, P_\mu = P(R)\nabla_\mu\phi$, in which case the Lagrangian and equations of motion become (after removal of a global multiplying constant)

$$-\mathcal{L}_{\mathrm{M}} = X'^2 + \beta\frac{P'^2}{R^2} + \frac{X^2P^2}{R^2} + \frac{1}{4}\left(X^2 - 1\right)^2 \quad (11)$$

and

$$X'' + \frac{X'}{R} - \frac{X^2P^2}{R^2} - \frac{1}{2}X\left(X^2 - 1\right) = 0 \quad (12a)$$

$$P'' - \frac{P'}{R} - \frac{1}{\beta}X^2P = 0, \quad (12b)$$

where a prime denotes differentiation with respect to R. To complement these equations, we must set the appropriate boundary conditions,

$$X(0) = P(\infty) = 0, \qquad X(\infty) = P(0) = 1. \quad (13)$$

With this we can solve the NO equations (12) numerically for various values of the parameter β. Sample solutions are plotted on figure 1.
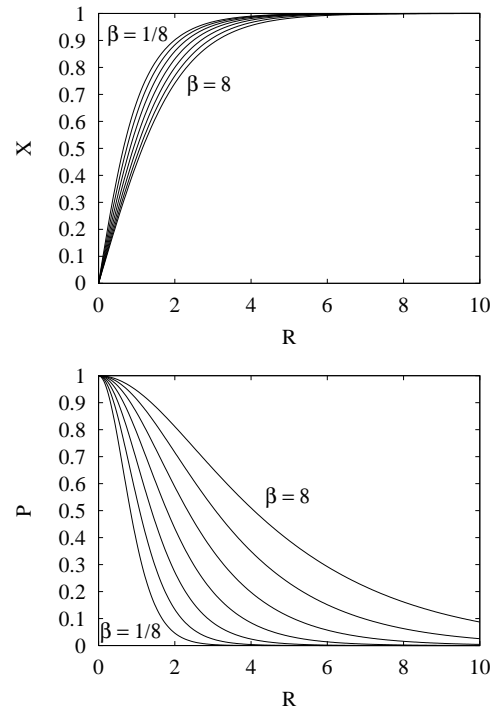

FIG. 1. The Nielsen–Olesen solution $X(R)$ and $P(R)$ for a few values of the Bogomoln'yi parameter, $\beta = 1/8, 1/4, 1/2, 1, 2, 4$ and 8.

The energy-momentum tensor for the NO solution is given from (11) by

$$T^\mu{}_\nu = \mathrm{Diag}\left(\rho, p_z, p_R, p_\phi\right), \quad (14)$$

where

$$\rho = \quad X'^2 + \beta\frac{P'^2}{R^2} + \frac{X^2P^2}{R^2} + \frac{1}{4}\left(X^2 - 1\right)^2 \quad (15a)$$

$$p_z = -\rho = \mathcal{L}, \quad (15b)$$

$$p_R = -X'^2 - \beta\frac{P'^2}{R^2} + \frac{X^2P^2}{R^2} + \frac{1}{4}\left(X^2 - 1\right)^2 \quad (15c)$$

$$p_\phi = \quad X'^2 - \beta\frac{P'^2}{R^2} - \frac{X^2P^2}{R^2} + \frac{1}{4}\left(X^2 - 1\right)^2. \quad (15d)$$

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Note that for $\beta = 1$ the equations of motion reduce to the Bogomol'nyi equations,

$$X' = \frac{XP}{R}, \qquad P' = \frac{R}{2}\left(X^2 - 1\right), \qquad (16)$$

which yields $p_R = p_\phi = 0$. For the purpose of finding the vortex's equation of state, we must integrate the energy and the pressures; these are shown in figure 2.
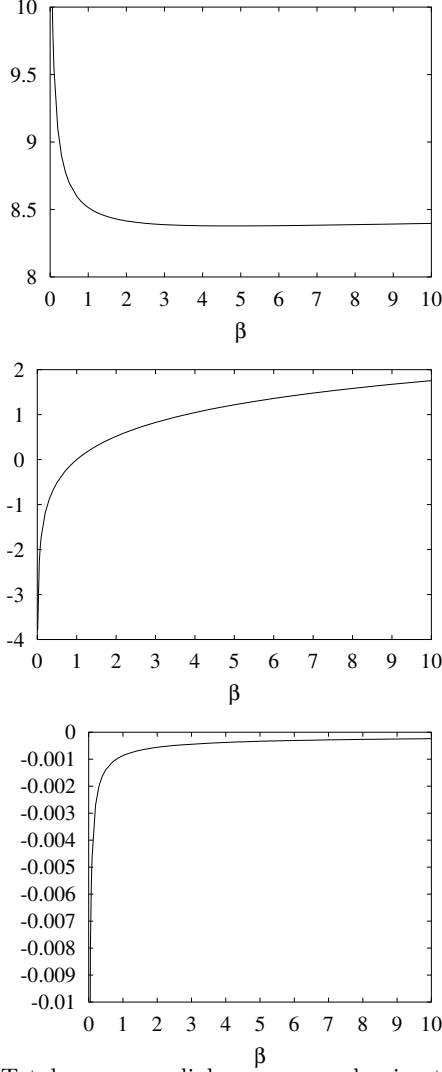

FIG. 2. Total energy, radial pressure and azimuthal pressure (top to bottom) for the NO solution, as functions of β.

Finally, we can write the equations of state of the vortex as

$$p_R = \alpha_R(\beta)\rho \qquad (17a)$$
$$p_\theta = \alpha_\theta(\beta)\rho. \qquad (17b)$$

The functions $\alpha_R(\beta)$ and $\alpha_\theta(\beta)$ are given on figure 3.

We are now in a position to determine realistic values for α_R and α_θ. Because the Abelian–Higgs theory is a toy model, there is no experimental determination of the

coefficients λ and e appearing in the definition of β, but we can compute it from the masses of the Higgs and the gauge boson. We assume here that

$$77 < m_H(\text{GeV}) < 300 \qquad (18)$$

and that the most probable value is $m_H \approx 170$ GeV [9]. For the gauge boson, we assume a $W^\pm$ or a Z^0, with

$$m_W = 80\text{GeV}, \qquad m_Z = 82\text{GeV}. \qquad (19)$$

This yields

$$0.88 \lesssim \quad \beta \quad \lesssim 14.0 \qquad (20a)$$
$$-0.011 \lesssim \quad \alpha_R \lesssim 0.239 \qquad (20b)$$
$$-1.103 \times 10^{-4} \lesssim \quad \alpha_\theta \lesssim -2.51 \times 10^{-5}, \qquad (20c)$$

the most probable value for m_H implying $\beta = 4.4$, $\alpha_R = 0.134$ and $\alpha_\theta = -4.27 \times 10^{-5}$.

We can now apply what we found to a spherical domain with metric (5) containing a fluid of vortices, by writing the equation of state for this fluid as

$$p = \frac{1}{2}(\alpha_R + \alpha_\theta)\rho = \kappa\rho, \qquad (21)$$

where $-5.51 \times 10^{-3} \lesssim \kappa \lesssim 0.119$ and most probable value $\kappa = 0.067$. Insertion of this into Soleng's result [7], via $\kappa = -1/\tilde{\alpha}$, gives

$$e^{2\nu} = e^{-2\lambda} = 1 - \frac{2M}{r} + \frac{1}{2\kappa + 1}\left(\frac{r}{r_0}\right)^{2\kappa} \qquad (22a)$$

$$\rho = -\frac{1}{r^2}\left(\frac{r}{r_0}\right)^{2\kappa}, \qquad (22b)$$

which yields, in the Newtonian limit, an effective acceleration of magnitude

$$g = \frac{M}{r^2} + \frac{K}{r^{1-2\kappa}}, \qquad (23)$$

where K is a constant. Numerically, we obtain that the correction decays like $r^{-1.011}$ (for $\beta = 0.88$), $r^{-0.866}$ (for $\beta = 4.4$) and $r^{-0.762}$ (for $\beta = 14$). It is exactly Milgrom ($\propto 1/r$) if $\beta = 1$.

Lastly, we compute the rotation curves for our modified dynamics, assuming a modified Hubble profile [10] to find $M(R)$. In figure 4 we plot the rotation curves for $K = 0.1$ and several values of β, as well as the curves for $\beta = 4.4$ for several values of K. The figure shows that, within the current limits on the Higgs mass, the modified dynamics allows for a broad family of curves, depending on the parameter K. Milgrom-type corrections are also allowed, although an exactly Milgrom $1/r$ dependence seems rather unlikely.

Although more tests would clearly be needed, the existence of a model such as the one presented in this Letter shows that one could imagine a situation, compatible with observations of the dynamics of galaxies and galaxy
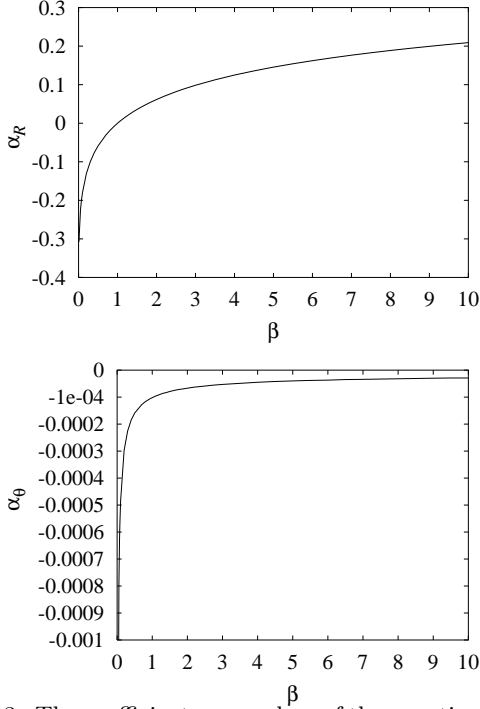
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FIG. 3. The coefficients α_R and α_θ of the equations of state for the NO vortex, as functions of β.

clusters, where dark matter would not be needed at all (or, taking a different point of view, would consist in our case of a fluid of vortices). This would have the advantage, over other dark matter models, that it is based on a particle theory (albeit a toy model here), giving it a stronger theoretical base.

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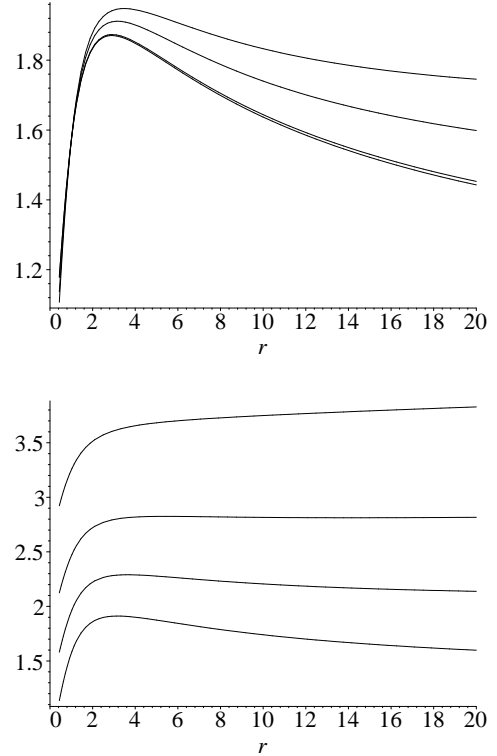

FIG. 4. Sample rotation curves obtained from the acceleration (23). On the top figure, we plot the curves for (bottom to top) $\beta = 0.88$, 1, 4.4 and 14. On the bottom figure, we plot the curves for $\beta = 4.4$ and (bottom to top) $K = 0.1$, 0.25, 0.5 and 1.

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